Realizability of Similar Formation and Local Control of Directed Multi-Agent Networks in Discrete-Time

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Abstract—This paper introduces a new concept called similar formation and studies the realizability condition, concerning with the existence of a protocol that has the ability to drive a directed multi-agent network to a desired formation shape. Based on the analysis of structure properties of complex Laplacian as opposed to real Laplacian, it is shown in the paper that in order to uniquely realize a similar formation in the plane, a necessary and sufficient graphical condition is that the directed multi-agent network is 2-rooted, a new type of connectivity. To the best of our knowledge, it is the first time to obtain a complete solution for the problem. On the other hand, a distributed control law is also derived using local measurements for the purpose of stabilizing a directed multi-agent network to a desired formation shape, which ensures globally asymptotic stability of the closed-loop system. Simulation results are provided as well to illustrate our results.

I. INTRODUCTION

In recent years, multi-agent systems (MAS) have attracted much research attention from various disciplines of engineering and science due to their interdisciplinary nature and broad potential applications. Collective behaviors arise from simple local interaction rules to permit sophisticated cooperation of the group that would never be achieved by individual members. Current research on MAS aims to provide deep understanding of group coordination and cooperation and inspire the design of distributed autonomous multi-agent systems.

In the paper, we consider the fundamental problems in formation control. What is the realizability condition for a formation shape in the plane and how to stabilize a directed multi-agent network to a desired formation shape globally using a distributed and linear control law? To this end, this paper aims to explore the novel topic of similar frameworks under only shape constraints, and exploit the design of global stabilizers.

For a given formation shape, a linear constraint defined on each agent can be derived, which sums up rotated and scaled relative states of its neighbors to zero. The composite constraints on all agents then specify a similar framework in terms of complex Laplacian of the directed multi-agent network. It is shown in our earlier work [17] that if and only if the complex Laplacian has two zero eigenvalues, the composite constraints lead to a desired formation shape up to rotation, translation and scaling, which is similar to the consensus subspace that can be specified by a linear constraint in terms of a real-valued Laplacian when it has a simple zero eigenvalue.

In this paper, we show that an equivalent graphical condition for the complex Laplacian to have two zero eigenvalues is that the graph is 2-rooted, a kind of new connectivity. That is, there exists a subset consisting of two nodes, from which every other node is 2-reachable. Consequently, a necessary and sufficient graphical condition for uniquely realizing a formation shape in the plane is the 2-rooted connectivity in the directed multi-agent network. To the best of our knowledge, it is the first time to obtain a complete graphical characterization for the realizability of a formation shape in directed multi-agent networks.

In addition, based on the complex Laplacian representation for a similar formation, a distributed and linear control law is then derived naturally using only relative state measurements without any global information. As a counterpart to our previous work [17], this paper concentrates on the discrete time setup. It is obtained without a surprise that the proposed local control law also ensures globally asymptotic stability as its continuous-time counterpart if a proper diagonal stabilizer can be found. To overcome the deficiency that the Newton iteration method may not provide a solution with a “bad” initial condition, the paper develops a homotopy Newton iteration method for the design of a proper diagonal stabilizer that can assign the eigenvalues at desired locations to have fast convergence rate.

The contribution of the paper is three-fold.

First, the paper introduces the control problem of achieving a similar formation, which is interesting itself with the objective of steering a team of mobile agents into a formation of variable size when the shape description of the formation is known but the desired size scaling of the formation is not known. Moreover, under the same framework, if additionally there are two or more agents who know the desired size scaling and can control themselves to achieve the desired distance between them, the whole team can then achieve a rigid formation even though the other team members do not have any knowledge about the size scaling and still follow the same linear interaction rule. Compared to existing techniques for rigid formation control ( [1], [7], [8], [11], [15], [20]), the sensing graph requires less links and also it is quite flexible to scale the size of the formation when not all agents but only a portion of agents in the network realize such necessity in certain situations.

Second, to our best knowledge, it is the first work providing global stability analysis for shape control of generic for-
mation and generic topology using distributed linear control laws. As pointed out in [5], [6], the desired formation shape specified by the distance constraint can be globally asymptotically stable under the gradient control law if and only if the formation graph is a tree, a particular structure of graphs. For more recent work on angle-based formation control, globally asymptotic stabilization for a formation shape is still challenging for a generic topology though some control laws ([2], [10], [14]) work for particular topology or particular formations such as triangular or circular formations.

Third, the work provides an original analysis for understanding the relationship between complex graph Laplacians and graphical connectivity in directed multi-agent networks. When a graph is undirected, a necessary and sufficient condition is presented in [18] to link the algebraic condition and the graphical condition for realizability of a similar formation in the plane. However, when a graph is directed, it is more difficult to analyze the relationship between complex graph Laplacians and graphical connectivity and thus only a necessary graphical condition is obtained for the realizability of a similar formation. This paper shows that the necessary graphical condition is also sufficient for almost all desired configurations of the directed multi-agent networks.

The organization of the paper is as follows. We review some knowledge of graph theory and introduce the problems in Section II. In Section III-A a necessary and sufficient graphical condition is obtained for the realizability of a similar formation. A distributed control law in discrete time is proposed Section III-B together with analysis of globally asymptotic behaviors. Simulation results are given in Section IV. Section V concludes our work and discusses several open problems.

II. PRELIMINARY AND PROBLEM SETUP

A. Notation

\( \mathbb{C} \) and \( \mathbb{R} \) denote the set of complex and real numbers, respectively. \( i = \sqrt{-1} \) denotes the imaginary unit. For a complex number \( p \in \mathbb{C}, |p| \) represents its modulus. For a set \( E \), \( |E| \) represents the cardinality. \( \mathbf{1}_n \) represents the \( n \)-dimensional vector of ones and \( I_n \) denotes the identity matrix of order \( n \).

B. Graph theory

A directed graph \( G = (V,E) \) consists of a non-empty node set \( V = \{1,2,\cdots,n\} \) and an edge set \( E \subseteq V \times V \). An edge of \( G \) is denoted by a pair of nodes \((j,i)\), which means that node \( i \) can access the relative positions between them. Throughout the paper, we let \( N_i \) denote the neighbor set of node \( i \), i.e., \( N_i = \{j : (j,i) \in E\} \). In the paper, we assume that a directed graph does not have self-loops, which means \( i \not\in N_i \) for any node \( i \).

Next, we introduce two concepts from [17], which are important to our development.

Definition 2.1: For a directed graph \( G \), a node \( v \) is said to be 2-reachable from a non-singleton set \( U \) of nodes if there exists a path from a node in \( U \) to \( v \) after removing any one node except node \( v \).

Definition 2.2: A directed graph \( G \) is said to be 2-rooted if there exists a subset of two nodes, from which every other node is 2-reaching. These two nodes are called roots in the graph.

Consider for example the graphs in Fig. 1. In Fig. 1(a), let \( U = \{u_1, u_2, u_3\} \) and it can be checked that node \( v \) is 2-reachable from \( U \) as after removing any one other node we are still able to find a path from a node in \( U \) to node \( v \). Thus for the graph in Fig. 1(a) \( v \) is 2-reachable from \( U \). In Fig. 1(b), \( v \) is not 2-reachable from \( U \). When we remove node \( u_2 \), there does not exist a path from a node in \( U \) to node \( v \). In Fig. 1(c), let \( U = \{u_1, u_2\} \) and it is known that the node \( v_1 \) is 2-reachable from the set \( U \) as after removing any one other node we are still able to find a path from a node in \( U \) to node \( v_1 \). Similarly, it is known that node \( v_2 \) and \( v_3 \) are also 2-reachable from \( U \) in Fig. 1(c). Therefore, the graph in Fig. 1(c) is 2-rooted.

![Examples of 2-reachable node, not 2-reachable node, and 2-rooted graph.](image)

Finally, we introduce a complex Laplacian for a directed graph. The complex-valued Laplacian \( L \) of a directed graph \( G \) is defined as follows: The \( i,j \)-th entry

\[
L(i,j) = \begin{cases} 
-w_{ij} & \text{if } i \neq j \text{ and } j \in N_i \\
0 & \text{if } i \neq j \text{ and } j \notin N_i \\
\sum_{j \in N_i} w_{ij} & \text{if } i = j 
\end{cases}
\]

where \( w_{ij} \in \mathbb{C} \) is called the complex weight on edge \((j,i)\). The definition of complex Laplacian is nothing different from real Laplacian except that the nonzero entries now can be a complex number. Consequently, it is also true that a complex Laplacian has at least one eigenvalue at the origin whose associated eigenvector is \( \mathbf{1}_n \) (namely, \( L\mathbf{1}_n = 0 \)).

C. Problem setup

Let \( G = (V,E) \) be a directed graph with \( n \) nodes. We denote a complex number \( \xi_i \in \mathbb{C} \) \((i = 1, \cdots, n)\) to represent a location in a reference frame \( \Sigma \). Define the \( n \)-dimensional composite complex vector \( \xi = [\xi_1, \xi_2, \cdots, \xi_n]^T \in \mathbb{C}^n \) a configuration for \( n \) agents in the reference frame \( \Sigma \). Throughout the paper, we assume that \( \xi_i \neq \xi_j \) if \( i \neq j \), meaning that no two agents overlap each other. More rigorously, a framework is defined as \((G,\xi,L)\) where \( G \) is a directed graph, \( \xi \) is a...
configuration and $L$ represents the linear constraint $L\xi = 0$ with $L$ being a complex Laplacian of $G$. Here a framework is defined in terms a linear constraint rather than the distance constraints on the edges of the graph as in [5], [6].

Definition 2.3: A framework $(G, \xi, L)$ is said to be similar if

$$\ker(L) = \{c_1 1_n + c_2 \xi : c_1, c_2 \in \mathbb{C}\}.$$  

Remark 2.1: A similar framework is a realization of the graph at certain points in the plane. We fix neither the reference coordinate frame nor the scale of the formation. Note that the complex number $c_2$ can be written in the polar coordinate form (namely, $c_2 = \rho e^{i\theta}$), from which we can see that the formation has four degrees of freedom: translation $c_1$, rotation $\theta$ and scaling $\rho$.

We consider a group of $n$ agents in the plane. The positions of $n$ agents are denoted by complex numbers $z_1, \ldots, z_n \in \mathbb{C}$. A directed graph $G$ of $n$ nodes represent the sensing graph in which an edge $(j, i)$ indicates that agent $i$ can measure the relative position between agent $i$ and $j$.

Suppose that the dynamics of agent $i$ is described by the following discrete-time equation:

$$z_i(k+1) = z_i(k) + u_i(k), \quad i = 1, \ldots, n.$$  

Denote the aggregate vector $z = [z_1, z_2, \ldots, z_n]^T \in \mathbb{C}^n$ and $u = [u_1, u_2, \ldots, u_n]^T \in \mathbb{C}^n$. Then the overall dynamics of the $n$ agents is

$$z(k+1) = z(k) + u(k). \quad (1)$$

In the paper we will study the following problems.

P1: What is the necessary and sufficient graphical conditions for the sensing graph $G$ such that there exists a local control law $u(k)$ driving the $n$ agents to achieve a similar formation in the plane?

P2: For a given graph $G$, how is a local control law $u(k)$ designed such that the $n$ agents asymptotically converge to a desired formation shape in the plane?

III. MAIN RESULT

In this section, we first characterize the realizability conditions for a similar framework in terms of graphical connectivity and second provides a distributed local control design for achieving a desired formation shape.

A. Realizability conditions of similar formations

For a network $G$, if a configuration $\xi$ of the $n$ agents is specified by a linear constraint related to the complex-valued Laplacian of the graph, a necessary and sufficient algebraic condition is presented in [17] for the realizability of a similar framework, which is re-stated in the following.

Theorem 3.1 ([17]): A framework $(G, \xi, L)$ is similar if and only if $\text{rank}(L) = n - 2$.

In this paper, we are going to find out the equivalent topological conditions characterizing the realizability of a similar framework. In [18], by assuming that the graph is undirected, a necessary and sufficient graphical condition is obtained. When $G$ is directed, it becomes more challenging as the zero and nonzero pattern of the Laplacian $L$ is not symmetric. Consequently, only a necessary graphical condition is given in [17]. We show in the following that it is also sufficient under some generic assumptions.

Theorem 3.2: A framework $(G, \xi, L)$ is similar for almost all $L$ and $\xi$ if and only if $G$ is 2-rooted.

Before providing the proof, we introduce a definition and give several lemmas.

Definition 3.1: The generic rank of a structured matrix $L$ is the maximal rank that $L$ achieves as a function of its arbitrary (non-zero) elements. Denote $\text{Grank}(L)$ as the generic rank of matrix $L$.

Lemma 3.1: If $G$ is 2-rooted, then its corresponding complex-valued Laplacian for arbitrary weights satisfies $\text{Grank}(L) \geq n - 2$.

Proof: Since $G$ is 2-rooted, we could find two nodes called roots in the graph and label them as $n - 1$ and $n$. Denote by $L_i$ the $i$-th leading principal sub-matrix of $L$. It is certain that there exists a complex Laplacian matrix $L$ whose diagonal entries are all nonzero. That is, all the first order principal minors of $L(n-2)$ are nonzero. Next we argue by induction that there exists an $L$ (a corresponding Laplacian for $G$) such that $\text{rank}(L(n-2)) = n - 2$. Suppose all the $(k-1)$-th order principal minors of $(L(n-2))$ are nonzero.

Consider a node subset $W$ with $|W| = k$ and denote by $W$ the sub-matrix of $L(n-2)$ formed by the rows and columns corresponding to the set $W$. Denote $\bar{W}$ as the node subset of the remaining nodes in $G$. Here we have $W \cup \bar{W} = V$. Then there must exist a node $i \in \bar{W}$, which has at least one neighbor in $W$ for the reason that both two roots are in $\bar{W}$. Denote that the $m$-th row of $W$ is the row corresponding to node $i$. Then according to the Laplace’s formula

$$\text{det}(W) = W(m, m)W_{mm} + \sum_{j=1,j\neq m}^{k} (-1)^{m+j}W(m, j)W_{mj}$$

where $W(m, j)$ is the $(m, j)$-th entry of $W$ and $W_{mj}$ is the determinant of the $(k-1) \times (k-1)$ matrix that results from $W$ by removing the $m$-th row and the $j$-th column. Note that $\text{det}(W) = 0$ if and only if $[W(m, 1), \cdots, W(m, k)]$ is in the orthogonal space of $[W_{m1}, \cdots, W_{mk}]$. As $G$ is 2-rooted, node $i$ has at least two neighbors. Besides, node $i$ has at least one neighbor in $W$. Therefore, we can arbitrarily choose nonzero entries in the $m$-th row of $W$ so that $\text{det}(W) \neq 0$ though it should satisfy the constraint $L1_n = 0$. By this way, it is proven that all $k$-th principal minors of $L(n-2)$ are nonzero. Thus, it follows from the induction argument that there exists an $L$ satisfying $\text{rank}(L(n-2)) = n - 2$, that is, $\text{Grank}(L) \geq n - 2$.

In the following, we will present a parameter representation of $L$ for a given 2-rooted graph $G$ and a configuration $\xi$.

As we have already known, any node of a 2-rooted graph has at least two neighbors. In the following, we consider two situations.

1) Consider the situation that all the nodes of $G$ have exactly two neighbors. We construct a new weighted graph called instantaneous graph, which has the same topology as $G$
but whose weights on the edges are defined according to the following rules: if node $i$ has two neighbors $j$ and $k$, then the weight on $(j, i)$ is $\xi_k - \xi_i$ and the weight on $(k, i)$ is $\xi_i - \xi_j$ (see Fig. 2 for an example). Then we define a Laplacian

\[ L = M_1(\xi) + \cdots + M_h(\xi), \]

where $P_i = \text{diag}\{p_{1i}, \cdots, p_{ni}\}$ with $p_{ij} \in \mathbb{C}$.

Now we are ready to introduce the notion of a regular configuration.

**Definition 3.2:** For a 2-rooted graph $G$, a configuration $\xi$ is called regular if there exists an instantial graph $G_\xi$ such that $\text{rank}(M_1(\xi)) = n - 2$. Otherwise, the configuration $\xi$ is called singular.

The second lemma comes from [13] without a proof.

**Lemma 3.2 ([13]):** For given matrices $A_1, \cdots, A_k$, if the determinant of matrix $A(x) = x_1A_1 + \cdots + x_kA_k$ is not zero for a choice of variables $x_1, \cdots, x_k$, then it is not zero for almost all choices of variables.

Here comes the last lemma about singular configurations.

**Lemma 3.3:** For a 2-rooted graph $G$, the singular configurations $\xi$ construct a set with measure zero.

**Proof:** Since $G$ is 2-rooted, it must be true that at least one instantial graph, saying without loss of generality $G_\xi$, is also 2-rooted. Then from Lemma 3.1, we know that there must be a Laplacian for $G_\xi$, denoted as $L^\ast$, satisfying $L^\ast \eta = 0$. Then we can calculate a vector $\eta$ satisfying $L^\ast \eta = 0$, with $\eta$ linearly independent of $1_n$. That means, $\text{rank}(M_1(\eta)) = n - 2$. Then by Lemma 3.2, it follows that for almost all $\xi$ $\text{rank}(M_1(\xi)) = n - 2$ (i.e., the configurations $\xi$ construct a set of measure zero).

**Proof of Theorem 3.2 (Sufficiency)** If $G$ is 2-rooted, then it follows from Lemma 3.1 that any Laplacian $L$ of $G$ satisfies $\text{rank}(L) \geq n - 2$. On the other hand, according to Definition 3.2 and Lemma 3.3, we know that for almost all $\xi$, there exists an instantial graph $G_\xi$ such that $\text{rank}(M_1(\xi)) = n - 2$. This means, there exists a Laplacian $L$ of $G$ satisfying $L\xi = 0$, which also makes the rank condition $\text{rank}(L) = n - 2$ hold. Therefore, by Lemma 3.2, it follows that for almost all Laplacians $L$ satisfying $L\xi = 0$, it holds that $\text{rank}(L) = n - 2$. Thus, the conclusion follows from Theorem 3.1.

**(Necessity)** The proof is given in [17].

Next, we give an example with a particular configuration $\xi$ such that even though $G$ is 2-rooted, the framework $(G, \xi, L)$ is not similar. However, it should be noted that such configurations are of measure zero as shown in the above results.

The example has a graph shown in Fig. 4, which is 2-rooted with node 5 and 6 being the two roots. Consider a configuration $\xi = [1 + \iota, -6\iota, -4/5 - 8/5\iota, -1 - \iota, -3 - 3\iota, 0]$. It can be checked that for any Laplacian $L$ of $G$.
satisfying $L\xi^{\prime} = 0$, rank($L$) = 3 < 4, which means there is another configuration $\eta$ that satisfies $L\eta = 0$ but is not similar to $\xi^{\prime}$.

However, arbitrarily perturbing the configuration $\xi^{\prime}$, for example, we take $\xi^{\prime\prime} = [1, -0.5 - 2.5t, -0.3 - 0.1t, -1 - t, 3 - t, 0.5]$. We are able to find a Laplacian $L''$ as follows

$$
\begin{bmatrix}
2 + 24t & 13 + t & -15 - 25t & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -t & 1 + t \\
0 & 0 & 2 + t & -1 & 0 & 2 - t \\
-4 & 0 & 0 & 2 - t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

satisfying $L''\xi^{\prime\prime} = 0$ and rank($L''$) = 4. Indeed, for all Laplacians $L$ satisfying $L\xi^{\prime\prime} = 0$ in this case, the rank condition rank($L$) = 4 holds and thus the framework $(G, \xi^{\prime\prime}, L)$ is similar.

**Remark 3.1:** From the analysis above, we know that the necessary and sufficient algebraic condition for a multi-agent system to achieve a similar formation is that the rank of the complex Laplacian $L$ of $G$ is $n - 2$. And the necessary and sufficient graphical condition is that the graph $G$ is 2-rooted. It is the first time to completely prove necessary and sufficient (algebraic and graphical) conditions for realizability of similar frameworks in the plane.

**B. Local control design for shape control**

The target formation shape is described by the subspace spanned by the desired configuration $\xi$ and $1_n$. We assume that the sensing graph of $n$ agents is 2-rooted. It is clear that the agents achieve the desired formation shape when the position vector $z$ converges to a point in $\{c_11_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$. Our goal in this section is to design a discrete-time local control law $u_i(k)$ using only relative position measurements of its neighbors so that

1. every point $z^*$ in $\{c_11_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$ is a stable equilibrium,
2. for any initial condition $z(0)$, the closed-loop trajectory approaches to an equilibrium in $\{c_11_n + c_2\xi : c_1, c_2 \in \mathbb{C}\}$.

The problem is referred to as a **global stabilization problem for shape control**.

Next we propose a distributed local control law in discrete time to solve the global stabilization problem for shape control. The control law is given in the form for individual agents, i.e.,

$$u_i(k) = d_i \sum_{j \in N_i} w_{ij}(z_j(k) - z_i(k)), \quad i = 1, \cdots, n, \tag{2}$$

where $w_{ij}$ is the complex weight selected to satisfy $L\xi = 0$ for purpose of shape control and $d_i \in \mathbb{C}$ is a control parameter to be designed later for the purpose of stabilization.

It is worth to be pointed out that $w_{ij}$ can be any complex number to make the corresponding Laplacian $L$ satisfy $L\xi = 0$, which is not unique. Preceding subsection also discusses a way in constructing $w_{ij}$’s from instantial graphs for a given desired configuration $\xi$. A simple rule is that the chosen complex weights make the relative state vectors of each agent’s neighbors rotated and scaled so that the summation becomes 0 at the desired formation shape. Take Fig. 5 as an example. Node $i$ has two neighbors (namely, $k$ and $j$). So the complex weights $\omega_{ij}$ and $\omega_{ik}$ rotate and scale the relative states $\xi_j - \xi_i$ and $\xi_k - \xi_i$ respectively so that the summation is zero as shown in Fig. 5.

![Fig. 5. An illustration of the distributed local control law.](image)

Next we introduce a property about the complex Laplacian $L$ of a 2-rooted graph.

**Lemma 3.4:** If $G$ is 2-rooted, for almost all $\xi$ and complex weights $w_{ij}$’s satisfying $L\xi = 0$, all the principal minors of $L$ up to $(n - 2)$-th order corresponding to the nodes in $V$ excluding the two roots are nonzero.

**Proof:** Without loss of generality, label the two roots of $G$ as $n - 1$ and $n$. If the graph is 2-rooted, according to the proof of Lemma 3.1, we can find a complex Laplacian $L^*$ whose principal minors from 1-th order to $(n - 2)$-th order are nonzero. Then, with the similar analysis as in the proofs for Lemma 3.3 and Theorem 3.2, we can obtain that all the principal minors of $L$ from 1-th order to $(n - 2)$-th order are nonzero for almost all $\xi$ and almost all complex weights $w_{ij}$’s.

**Remark 3.2:** As discussed above, the complex weights $w_{ij}$’s in (2) are randomly chosen to satisfy $L\xi = 0$ and thus it owns the property stated in Lemma 3.4.

With local individual control law (2), the overall closed-loop dynamics of $n$ agents becomes

$$z(k + 1) = (I - DL)z(k) \tag{3}$$

where $D = \text{diag}(d_1, \cdots, d_n)$ is an $n$-by-$n$ diagonal complex matrix.

To stabilize the system to a desired formation shape, it is important to find a proper $D$ so that $I - DL$ has eigenvalues strictly inside the unit disk centered at the origin in addition to two fixed eigenvalues at (1, 0). We refer to $D$ a **stabilizing matrix** if it is able to re-assign the eigenvalues of $DL$ inside the unit circle centered at (1, 0) in addition to two fixed eigenvalues at the origin. The following is the result for the existence of a stabilizing matrix $D$.

**Theorem 3.3:** For a framework $(G, \xi, L)$, if a graph $G$ is 2-rooted, then for almost all desired configuration $\xi$, a stabilizing matrix $D$ exists and moreover it can assign the eigenvalues of the closed-loop system (3) at any location in addition to two fixed eigenvalues.

The proof requires a result related to the multiplicative inverse eigenvalue problem by Friedland in 1975.
Lemma 3.5 ([9]): Let $A$ be an $n \times n$ complex-valued matrix. Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be an arbitrary set of $n$ complex numbers. If all principal minors of $A$ are distinct from zero, then there exists a diagonal complex valued matrix $D$ such that the spectrum of $DA$ is the set $\sigma$. Moreover, the number of different matrices $D$ is at most $n!$.

Proof of Theorem 3.3: Without loss of generality, label the two roots of a 2-rooted graph $\mathcal{G}$ as $n - 1$ and $n$. According to Remark 3.2, all principal minors of $L_{n-2}$ are nonzero for $L$ satisfying $L^*_L = 0$.

Therefore, by Lemma 3.5, there exists a diagonal complex matrix $D_1$ arbitrarily assigning the eigenvalues of $D_1 L_{n-2}$, which implies, a stabilizing matrix $D$ exists and the eigenvalues of the closed-loop system (3) can be assigned at any locations in addition to the two fixed eigenvalues.

Next we are going to present an algorithm on how to design a stabilizing matrix $D$ such that the eigenvalues of the closed-loop system (3) lie exactly at $\sigma = \{\lambda_1, \ldots, \lambda_{n-2}, 1, 1\}$. The desired eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$ can be chosen according to additional performance requirements.

Since $L$ has rank $n - 2$, it then follows that $L$ can be factorized into $L = UV$ where $U \in \mathbb{C}^{n \times (n-2)}$ and $V \in \mathbb{C}^{(n-2) \times n}$ satisfy $\text{rank}(U) = \text{rank}(V) = n - 2$. Notice that

$$\det(sI - DL) = \det(sI - DUV) = s^n \det(sI - VDU).$$

The problem of designing $d_i$ ($i = 1, \ldots, n$) such that the spectrum of $I - DL$ is the set $\sigma$ is equivalent to the problem of finding $d_i$ ($i = 1, \ldots, n$) such that $VDU$ has eigenvalues at

$$\{-\lambda_1 + 1, \ldots, -\lambda_{n-2} + 1\}.$$

Theorem 3.3 ensures the existence of $D$, and in a generic sense there are infinite number of solutions for the above problem. So we could arbitrarily assign two values to two $d_i$’s. Without loss of generality, select $d_{n-1}$ and $d_n$ and set $d_{n-2} = d_a = 1$. Denote $\tilde{d} = (d_1, d_2, \ldots, d_{n-2})$ and denote $A(\tilde{d}) = VDU$ with $d_{n-1} = d_n = 1$. Generically, there are $(n-2)!$ solutions of $\tilde{d}$ to assign the eigenvalues of $A(\tilde{d})$ at

$$\{-\lambda_1 + 1, \ldots, -\lambda_{n-2} + 1\}.$$

In this paper, we propose a homotopy Newton iteration method to solve $\tilde{d}$. In [12], we consider a Newton iteration method to solve $\tilde{d}$. But finding a good starting value for the Newton’s method is a crucial problem. The homotopy method can be used to generate a good starting value.

Suppose $F(x) = 0$ is the function we want to solve. Assume that $F_0$ is a known function with a known zero $x^*$, i.e., $F_0(x^*) = 0$. We construct a parameter depending function

$$H(x, s) = s F(x) + (1 - s) F_0(x) \quad s \in [0, 1]$$

where $H(x, 0) = 0$ is the problem with known solution and $H(x, 1) = 0$ is the original problem $F(x) = 0$.

An example for $F_0(x)$ is just the trivial way

$$F_0(x) := F(x) - F(x^*).$$

This gives the homotopy function

$$H(x, s) = F(x) + (s - 1) F(x^*), \quad (x^* \text{ is a given vector}).$$

As the solution of $H(x, s) = 0$ depends on $s$, we denote it by $x^*(s)$. We discretize now the interval into $0 = s_0 < s_1 < \cdots < s_n = 1$ and solve a sequence of nonlinear systems with the Newton’s method

$$H(x, s_i) = 0.$$

Each iteration starts with the solution $x^*(s_{i-1})$ of the preceding problem.

Define

$$F(\tilde{d}) = \begin{bmatrix}
\det(A(\tilde{d}) + \lambda_1 I - I) \\
\vdots \\
\det(A(\tilde{d}) + \lambda_{n-2} I - I) \\
\end{bmatrix} = \begin{bmatrix}
F_1(\tilde{d}) \\
\vdots \\
F_{n-2}(\tilde{d}) \\
\end{bmatrix}$$

where $\det(\cdot)$ represents the determinant of a matrix. Clearly, $\tilde{d}$ is a solution of the eigenvalue assignment problem if and only if $F(\tilde{d}) = 0$. Denote

$$H(\tilde{d}, s) = F(\tilde{d}) + (s - 1) F(\tilde{d}^*), \quad (\tilde{d}^* \text{ is a given vector}) \quad (5)$$

$\tilde{d}$ is a solution of the problem if and only if $H(\tilde{d}, 1) = 0$. In order to apply a Newton iteration process on $F$ we need the partial derivatives of $H(\tilde{d}, s)$ with respect to $\tilde{d}$:

$$g_{ij} = \frac{\partial H_i(\tilde{d}, s)}{\partial d_j} = \frac{\partial F_i(\tilde{d})}{\partial d_j}, \quad G(\tilde{d}) = (g_{ij}) \quad i, j = 1, \ldots, n - 2.$$

Using the Trace-Theorem of Dacidenko [4], we get, if $F_i(\tilde{d}) \neq 0$,

$$g_{ij} = \frac{\partial F_i(\tilde{d})}{\partial d_j} = F_i(\tilde{d}) \cdot \text{tr}\left((F_i(\tilde{d}))^{-1} \cdot \frac{\partial (F_i(\tilde{d}))}{\partial d_j}\right)$$

$$= F_i(\tilde{d}) \cdot \text{tr}((F_i(\tilde{d}))^{-1} \cdot : V(:, j) \cdot U(:, i)).$$

Thus, the solution can be solved iteratively as follows:

$$\tilde{d}^{m+1} = \tilde{d}^m - G(\tilde{d}^m)^{-1} H(\tilde{d}^m, s), \quad m = 0, 1, 2, \ldots$$

IV. SIMULATIONS

In this section, we will illustrate our result through simulations.

We consider a group of 5 agents. The desired configuration $\xi$ (formation shape) is depicted in Fig. 6, for which

$$\xi = [\imath^T \ 1 + 2\imath \ - 1 + 2\imath \ - 1] \in \mathbb{C}$$

where $\imath$ is the imaginary unit.

The sensing graph $\mathcal{G}$ is given in Fig. 7. It can be checked that $\mathcal{G}$ is 2-rooted because $\{3, 4\}$ can be chosen as the subset of two roots and every other node is 2-reachable from $\{3, 4\}$. However, for the graph shown in Fig. 7, it can swing around node 1 if only the distance constraints are imposed, which means, such a formation shape cannot be maintained using the control laws based on graph rigidity in literature [3], [10], [16], [19].

With our control scheme, the desired formation shape can be achieved. Arbitrary complex weights can be selected to
satisfy $L \xi = 0$, for example,

$$L = \begin{bmatrix}
-2\iota & 0 & 1 + \iota & -1 - \iota & 0 \\
2 & -1 + \iota & -1 - \iota & 0 & 0 \\
2 & -1 + \iota & -1 - \iota & 0 & 0 \\
-2 & 0 & 0 & 1 - \iota & 1 + \iota \\
-2 & 0 & 0 & 1 - \iota & 1 + \iota
\end{bmatrix}.$$ 

It is not a surprise that $\text{rank}(L) = 3$. By checking the eigenvalues of $L$, it is known that the distribution of the eigenvalues of $L$ cannot ensure stability of system (3). By assigning the desired eigenvalues of $I - DL$ at

$$\sigma_1 = \{0.25 - 0.25\iota, 0.75 + 0.25\iota, 0.75, 1, 1\}$$

and

$$\sigma_2 = \{-0.5 - 0.5\iota, 0.5 + 0.5\iota, 0.5, 1, 1\}$$

respectively, we solve stabilizing matrices using the homotopy Newton iteration method, which are

$$D_1 = \text{diag}\{ -0.5059 - 0.1740\iota, 0.1779 + 0.5061\iota, 0.2008 + 0.4828\iota, 1, 1\}$$

and

$$D_2 = \text{diag}\{ -0.3451 + 0.0324\iota, -0.4784 + 0.2301\iota, -0.1025 + 0.0843\iota, 1, 1\}.$$ 

With the above two stabilizing matrices, the two simulation results are shown in Fig. 8 and Fig. 9. It can be foreseen that different eigenvalue assignments lead to different motions in achieving the desired formation shape but both ensure the asymptotic convergence towards the desired formation shape.

**V. CONCLUSIONS AND FUTURE WORK**

In the paper, we develop a necessary and sufficient graphical condition for a framework to be similar. That is, a graph describing the sensing relationship of networked multiple agents should be 2-rooted, with which for almost all desired formation shape, it is able to control networked multiple agents towards the desired formation shape using distributed linear control laws. Globally asymptotic stability analysis is presented and moreover, a procedure is provided to reassign the closed-loop eigenvalues so that more performance specification can be satisfied such as fast convergence rate.

In the paper we focus on the formation control problem.
of networked mobile agents in the plane. The methods, however, are general, and they have applicability beyond multi-vehicle formations, e.g., distributed beamforming of communication systems and power networks where consensus is not an objective but achieving a pattern is a goal. This work serves as a starting point for many problems in this framework. Future work includes the formation shape control for directed multi-agent networks with time-varying or stochastic topology, or for directed multi-agent networks with complicated individual dynamics.

References


