Local Formation Control Strategies with Undetermined and Determined Formation Scales for Co-leader Vehicle Networks

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Abstract—Formation control of multi-agent systems involves harmony between information flow networks and distributed local controller design. In this paper, we first consider a novel problem of controlling a group of agents with only shape constraints but with a flexible scale. Relative position sensing information and communication information of velocity estimates are represented in two separate graphs. A distributed local control strategy is obtained, in which each agent responds to nearby agents in the sensing graph for shape control and synchronizes its velocity to its neighbors in the communication graph. A group of agents then asymptotically converges to a desired formation shape in moving. Second, by assuming that the desired scale of the formation is known to two leaders of the network, the team of agents can then achieve a desired formation with a determined scale controlled by the two co-leaders while other team members still follow the same local control law for shape control. Finally, a collision avoidance scheme is introduced to avoid the occurrence of collisions between each other in the process of achieving the desired formation shape.

I. INTRODUCTION

Recent years, cooperative and coordinated control for multi-agent systems (MAS) has received significant attention from various disciplines of engineering and science, because of broad potential applications in both military and civil environments. Modeling the interaction topology of distributed agents as an undirected graph or a directed graph, graph Laplacians play an important role in the analysis of collective behaviors and the design of distributed local control laws.

This paper continues this effort by using complex-valued Laplacian as opposed to real-valued Laplacian for the study of formation control. Most works concern rigid formation control using different techniques such as consensus based schemes [6] [9], artificial potential approaches [8] [11], and rigid graph related formation control strategies [1]. This paper, however, concentrates on formation control with variable scaling. By allowing the scale of formation to change, a formation of agents can dynamically adapt to complicated environments in responding to the change of the environments. In such a case, it is desired that most agents in the network still follow their simple local control laws without a re-design and only a portion of foreseen agents may need to change their strategies for the purpose of regulating the scale of the whole formation. While most distance-based formation control strategies do need a re-design for all the agents in the network to vary the formation scale, [2] presents two strategies that allow the agents to maneuver to the desired scaled formation using only local relative position information. In addition, [7] introduces a novel idea based on complex-valued Laplacian to steer a group of agents to a desired formation shape with flexible scales. However, it deals with only the stationary formation stabilization problem. Later, [4] extends the idea for moving formation control with flexible scales, but it suffers a fatal deficiency of not able to compute the control signals in real time due to mutual dependence of control inputs in the network.

In this paper, we introduce an auxiliary dynamic system for each agent to store and update the estimate of its own velocity according to the information from its neighbors in the communication network. The formation control for each agent then utilizes this piece of information as well as responds to nearby agents in the sensing network using only relative position measurements. By using the result that a complex-valued Laplacian can be stabilized by a diagonal complex matrix [7], it is obtained that a network of agents can globally asymptotically converge to a desired formation shape while following the leaders in moving. On the other hand, in order to control or change the formation scale in responding to the environment change, we assume only two co-leaders in the network are aware of the distance specification between them. A local control law is borrowed from [3] to make the two co-leaders achieve the desired distance between them while all the other agents make no change for their control laws. Using the theory of input-to-state stability and asymptotic stability for cascade systems, we show that the whole network asymptotically converges to the desired formation with its scale controlled by the two co-leaders. Finally, motivated by the demand of avoiding collisions in formation control, additional control effort is introduced to make collisions between agents not able to happen, which takes the artificial potential approach.

Compared to [2], this paper presents a relatively simpler control design for formation shape control with the formation scale being controlled by two co-leaders and also provides a global asymptotic stability analysis. Moreover, this paper overcomes the fatal deficiency of the control law in [4] and to the best of our knowledge, it is the first result on the formation control problem for which the leaders control the formation scale while the followers control the formation shape using distributed linear control laws.

The organization of the paper is as follows. We provide some preliminary materials on directed graphs and problem setup in Section II. In Section III the formation control prob-
lems of achieving a moving formation with undetermined and determined scales as well as the collision avoidance problem are investigated. Simulations are presented in Section IV. Section V concludes our work and discusses open problems.

**Notations:** $\mathbb{C}$ and $\mathbb{R}$ denote the set of complex and real numbers, respectively. $i = \sqrt{-1}$ denotes the imaginary unit. $1_n$ represents the $n$-dimensional vector of ones and $I_n$ denotes the identity matrix of order $n$. $\{\lambda(A)\}$ denotes the set of all the eigenvalues of matrix $A$.

**II. PRELIMINARIES AND PROBLEM SETUP**

**A. Graph theory**

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty node set $\mathcal{V} = \{1, 2, \cdots , n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge of $\mathcal{G}$ is denoted by an ordered pair of nodes $(j,i)$, which means that the edge has tail at node $j$ and has head at node $i$. Alternatively, the edge $(j,i)$ is called an incoming edge of node $i$ and outgoing edge of node $j$. Throughout the paper, we let $\mathcal{N}_i$ denote the in-neighbor set of node $i$, i.e., $\mathcal{N}_i = \{ j : (j,i) \in \mathcal{E} \}$, and let $n_i$ denote the cardinality of $\mathcal{N}_i$. In the paper, we assume that a digraph does not have self-loops, which means $i \notin \mathcal{N}_i$ for any node $i$.

For a digraph $\mathcal{G}$, we associate to each edge $(j,i)$ a complex number $w_{ij} \neq 0$, called complex weight. Then we can define a complex-valued Laplacian $L$, for which

$$L(i,j) = \begin{cases} -w_{ij} & \text{if } i \neq j \text{ and } j \in \mathcal{N}_i \\ 0 & \text{if } i \neq j \text{ and } j \notin \mathcal{N}_i \\ \sum_{j \in \mathcal{N}_i} w_{ij} & \text{if } i = j. \end{cases}$$

If all $w_{ij}$ are real, it reduces to the real-valued Laplacian that we are familiar with.

**B. Geometric formation**

We now present several notions related to 2D geometric formations. In the plane, a tuple of $n$ complex numbers

$$\xi = [\xi_1, \xi_2, \cdots , \xi_n]^T$$

is called a formation vector for $n$ agents, which defines a geometric pattern in a specific coordinate system. Usually two agents are not supposed to overlap each other, so we assume that

$$\xi_i \neq \xi_j \quad \text{for } i \neq j.$$

A formation with four degree of freedom (translation, rotation, and scaling) is defined by

$$F_\xi = c_1(t)1_n + c_2(t)\xi,$$

where $c_1(t), c_2(t) \in \mathbb{C}$. As an example, the geometric formations $F_\xi^1$, $F_\xi^2$, and $F_\xi^3$ of four agents in Fig. 1 are obtained from the same basis via translating, rotating, and scaling. When $|c_2(t)| = 1$, then the formation is obtained from the formation vector via translation and rotation only (a rigid transformation), a case which we are more familiar with.

**C. Problem setup**

We consider a group of $n$ agents in the plane labeled $1, \ldots, n$, consisting of leaders and followers. Suppose that there are two leaders in the group (without loss of generality, say 1 and 2) and all the others are followers. The positions of the $n$ agents are denoted by complex numbers $z_1, \ldots, z_n \in \mathbb{C}$. We use a digraph $\mathcal{G}$ of $n$ nodes, which is called the sensing graph, to represent the sensing relationship of relative position measurements among the agents. Then we have the following assumption.

**Assumption 2.1:** Each agent can access the relative position information about its neighbors in the sensing graph.

In the sensing graph, $\{1, 2\}$ are leader agents, $\{3, \ldots, n\}$ are follower agents, and an edge $(j,i)$ indicates that agent $i$ can measure the relative position of agent $j$, namely, $(z_j - z_i)$. In the first case, we assume that the co-leaders in a co-leader vehicle network do not interact with others and do not need to access the information from others. Thus, the Laplacian of $\mathcal{G}$ takes the following form

$$L = \begin{bmatrix} 0 & \tilde{F} \\ \tilde{F} & 0 \end{bmatrix} \in \mathbb{R}^{2n \times (n-2)},$$

(1)

On the other hand, we use a digraph $\mathcal{H}$ of $n$ nodes, which is called the communication graph, to represent the velocity information exchange among the agents. Then, we have a similar assumption.

**Assumption 2.2:** Each agent can receive the velocity estimate from its neighbors in the communication graph.

The communication graph includes the two leaders and all the other followers, where an edge $(j,i)$ indicates that agent $i$ can receive the estimated velocity from agent $j$. Thus, the communication graph $\mathcal{H}$ may have different topology from the sensing graph $\mathcal{G}$. Assume the two leaders do not exchange information between them and do not receive communication information from the followers. Then the Laplacian of $\mathcal{H}$ also

![Fig. 1. Formations up to translation, rotation and scaling.](image-url)
takes the similar form as (1). That is,
\[
H = \begin{bmatrix}
0_{2 \times 2} & 0_{2 \times (n-2)} \\
H_{ff} & H_{ff}
\end{bmatrix}.
\] (2)

Here we consider a real-valued Laplacian for the communication graph $\mathcal{H}$. So its element takes the following form
\[
H(i, j) = \begin{cases}
-\alpha_{ij} & \text{if } i \neq j \text{ and } j \in N_i(\mathcal{H}) \\
0 & \text{if } i \neq j \text{ and } j \notin N_i(\mathcal{H}) \\
-\sum_{k \in N_i(\mathcal{H})} H(i, k) & \text{if } i = j
\end{cases}
\]
where $\alpha_{ij}$ is an arbitrary positive number and $N_i(\mathcal{H})$ is the in-neighbor set of agent $i$ for the communication graph $\mathcal{H}$.

In the paper we will study the following problems.

**Problem 1:** How to asymptotically achieve a moving geometric formation $F_\xi = (c_1 + v_0 t) \mathbf{1}_n + c_2(t) \xi$ with undetermined formation scale, where $v_0$ is the synchronized velocity of the co-leaders?

If additionally the co-leaders know the desired distance between them for the desired formation, then the formation scale should be able to be controlled by the co-leaders using their relative position measurement about each other. We aim to solve this problem, which is stated below.

**Problem 2:** How to asymptotically achieve a moving geometric formation with determined formation scale controlled by two co-leaders?

The agents of the system may collide each other in the process of achieving a moving geometric formation. Thus, the following problem is introduced.

**Problem 3:** How to avoid collisions between each other in the process of achieving a moving geometric formation?

### III. MAIN RESULTS

**A. Formation shape control with undetermined scale**

We assume that each agent is governed by a single integrator kinematics model
\[
\dot{z}_i = u_i,
\] (3)
where $z_i \in \mathbb{C}$ represents the position of agent $i$ in the plane and $u_i \in \mathbb{C}$ represents the velocity control input.

Consider that the two leaders take the synchronized constant velocity $v_0 \in \mathbb{C}$. The control law for the leaders is given as
\[
\dot{z}_i = v_0, \quad i = 1, 2.
\] (4a)

According to the sensing graph $\mathcal{G}$, each follower obtains its neighbors’ relative position, while according to the communication graph $\mathcal{H}$, each follower gets its neighbors’ estimated velocity. Hence, the following control law is proposed for each follower,
\[
\begin{align*}
\dot{z}_i &= \sum_{j \in N_i(\mathcal{G})} w_{ij}(z_j - z_i) + \eta_i, \quad i = 3, \cdots, n \\
\dot{\eta}_i &= \sum_{j \in N_i(\mathcal{H})} \alpha_{ij}(\eta_j - \eta_i)
\end{align*}
\] (4b)
where $w_{ij}$ is a complex weight of the complex-valued Laplacian $L$ such that $L \xi = 0$ for a desired formation vector $\xi$, $\alpha_{ij}$ is an arbitrary positive number of the real-valued Laplacian matrix $H$, $N_i(\mathcal{G})$ and $N_i(\mathcal{H})$ are the in-neighbor sets of the agent $i$ in the sensing graph $\mathcal{G}$ and the communication graph $\mathcal{H}$, and $\eta$ is an estimate of the velocity of agent $i$. Note that $\eta_1 = \eta_2 = v_0$ and $\dot{\eta}_1 = \dot{\eta}_2 = 0$ since $v_0$ is a constant velocity.

Let $z = [z_1, z_2, \ldots, z_n]^T \in \mathbb{C}^n$, and $\eta = [\eta_1, \eta_2, \ldots, \eta_n]^T \in \mathbb{C}^n$. Then the overall dynamics of the system can be described as
\[
\begin{bmatrix}
\dot{z} \\
\dot{\eta}
\end{bmatrix} =
\begin{bmatrix}
-L & I_n \\
0 & -H
\end{bmatrix}
\begin{bmatrix}
z \\
\eta
\end{bmatrix}
\] (5)
where $L$ is the complex-valued Laplacian of $\mathcal{G}$ defined in (1), $H$ is the real-valued Laplacian $\mathcal{H}$ defined in (2). First, we show a result that the steady-state solution of system (5) describes a moving geometric formation. Here we denote $\bar{z}_1 = z_1(0)$ and $\bar{z}_2 = z_2(0)$.

**Theorem 3.1:** Suppose $\xi \in \mathbb{C}^n$ satisfying $\xi_i \neq \xi_j$. The system (5) has a unique steady solution $z^*(t) = (c_1 + v_0 t) \mathbf{1}_n + c_2 \xi$ and $\eta^*(t) = v_0 \mathbf{1}_n$, with
\[
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = \begin{bmatrix} 1 & \xi_1 \\
1 & \xi_2 \end{bmatrix}^{-1}
\begin{bmatrix}
\bar{z}_1 \\
\bar{z}_2
\end{bmatrix}
\] (6)
if and only if
\[
L \xi = 0, \quad \det(L_{ff}) \neq 0 \quad \text{and} \quad \det(H_{ff}) \neq 0.
\]
The proof of the theorem requires the following lemma.

**Lemma 3.1 (4):** Suppose $\xi \in \mathbb{C}^n$ satisfying $\xi_i \neq \xi_j$. Consider system $\dot{z} = -ALz$ where $A$ is non-singular and $z_1(0) = \bar{z}_1$ and $z_2(0) = \bar{z}_2$ ($\xi_1 \neq \xi_2$). Then the system has a unique equilibrium state $z^* = c_1 \mathbf{1}_n + c_2 \xi$ with
\[
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = \begin{bmatrix} 1 & \xi_1 \\
1 & \xi_2 \end{bmatrix}^{-1}
\begin{bmatrix}
\bar{z}_1 \\
\bar{z}_2
\end{bmatrix}
\]
if and only if
\[
L \xi = 0 \quad \text{and} \quad \det(L_{ff}) \neq 0.
\]

**Proof of Theorem 3.1:** According to the definitions of complex-valued Laplacian matrix $L$ and real-valued Laplacian matrix $H$, we know $L \mathbf{1}_n = 0$ and $H \mathbf{1}_n = 0$. Let $y = z - v_0 t \mathbf{1}_n$ and $\delta = \eta - v_0 \mathbf{1}_n$. Then system (5) is transformed into
\[
\begin{bmatrix}
\dot{y} \\
\dot{\delta}
\end{bmatrix} =
\begin{bmatrix}
-L & I_n \\
0 & -H
\end{bmatrix}
\begin{bmatrix}
y \\
\delta
\end{bmatrix}
\] (7)
which has the same matrix form as (5). By the conditions in the theorem, $\det(H_{ff}) \neq 0$. So we know $\text{rank}(H) = n - 2$. Let $\ker(H) = \{a_1 \mathbf{1}_n + a_2 h\}$, where $a \in \mathbb{R}^n$ and $\mathbf{1}_n \in \mathbb{R}^n$ are two linearly independent eigenvector associated to the zero eigenvalue of both algebraic and geometric multiplicity 2. Denote
\[
\mathbf{1}_n = \begin{bmatrix} 1 \\ 1_{n-2} \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} h_1 \\ h_f \end{bmatrix},
\]
where $h_1 \in \mathbb{R}^2$ and $h_f \in \mathbb{R}^{n-2}$.

We claim that $\mathbf{1}_2$ and $h_1$ are linearly independent. To this end, we assume $\mathbf{1}_2$ and $h_1$ are linearly dependent, i.e., $h_1 =
valued Laplacian of $H$ asymptotically reaches a moving geometric formation as others may on both right and left complex plane. If $D$ has four zero eigenvalues. Thus, we obtain that the system
\[
\{ \begin{align*}
H_f I_2 + H_f f I_{n-2} &= 0 \\
H_f h_1 + H_f h_f &= 0. 
\end{align*} \tag{8}
\]
Substituting $h_1 = f I_2$ into (8), we then obtain that $h_f = f I_{n-2}$, which means that $I_n$ and $h$ are linearly dependent, a contradiction. So we have $I_2$ and $h_f$ are linearly independent.

Since $\delta_1 = \delta_2 = 0$, we obtain that $a_1 I_2 + a_2 h = 0_2$ at steady solutions, where $a_1, a_2 \in \mathbb{R}$. Thus, we have $a_1 = a_2 = 0$ and so it follows that at equilibrium $\delta^* = a_1 I_n + a_2 h = 0 \in \mathbb{R}^n$. Substituting $\delta$ into system (7), we get $\dot{y} = -Ly$. By lemma 3.1, it is obtained that if and only if $L \xi = 0$ and det$(L_{ff}) = 0$, system (7) has the equilibrium states $\dot{y} = c_1 I_n + c_2 \xi$ where $c_1, c_2$ can be calculated from (6).

Considering the original system of $z$ and $\eta$, then we obtain that the steady solution of (5) is the one given in the theorem, describing the moving geometric formation $F_\xi = (c_1 + c_2 t) I_n + c_2 \xi$.

Next we come to study the stability of the system.

**Theorem 3.2:** If a matrix $D_f \in \mathbb{R}^{n-2}$ can assign all the eigenvalues of $D_f L_{ff}$ in the right open complex plane, then the system
\[
\begin{bmatrix}
\dot{z} \\
\dot{\eta}
\end{bmatrix} = 
\begin{bmatrix}
-DL & I_n \\
0 & -H
\end{bmatrix}
\begin{bmatrix}
z \\
\eta
\end{bmatrix} \tag{9}
\]
asymptotically reaches a moving geometric formation as described in Theorem 3.1, with $D = \text{diag}(I_2, D_f)$.

**Proof:** We know $L$ has two zero eigenvalues and the others may on both right and left complex plane. If $D_f$ can assign all the eigenvalues of $D_f L_{ff}$ in the right open complex plane, then $D$ can assign the eigenvalues of $DL$ in the right complex plane in addition to two fixed zero eigenvalues.

Let
\[
T = 
\begin{bmatrix}
-DL & I_n \\
0 & -H
\end{bmatrix}
\]
Since $T$ is a block upper triangular matrix, the set of all the eigenvalues of $T$, denoted by $\{\lambda(T)\}$ satisfies $\lambda(T) = \{\lambda(-DL) \cup \lambda(-H)\}$. According to the discussion above, $-DL$ has two zero eigenvalues and all the other eigenvalues are in open left complex plane. Moreover, since $H$ is a real-valued Laplacian of $\mathcal{H}$, all the eigenvalues (i.e., $\lambda(-H)$) are in the left complex plane except two fixed zero eigenvalues.

With the above discussion, the system (9) has all eigenvalues are in the open left complex plane in addition to four zero eigenvalues. Thus, we obtain that the system (9) asymptotically reaches a moving geometric formation according to Theorem 3.1.

**Remark 3.1:** From [7], we know that if det$(L_{ff}) \neq 0$, then $D_f$ exists almost surely, which can re-assign the eigenvalues of $D_f L_{ff}$ at any desired locations. Therefore, by Theorem 3.2, the following control law modified from (4b)
\[
\begin{align*}
\dot{z}_i &= \sum_{j \in N_i(G)} w_{ij} (z_j - z_i) + \eta_i, \\
\dot{\eta}_i &= \sum_{j \in N_i(\mathcal{H})} \alpha_{ij} (\eta_j - \eta_i), \quad i = 3, \cdots, n
\end{align*} \tag{10}
\]
where $d_i$ is the $i$th diagonal entry of $D$, can be used to control a group of agents towards the desired formation shape with its scale relying on the initial condition.

**Remark 3.2:** If the velocity of two co-leaders are unsynchronized, a consensus law can be adopted for the co-leaders to asymptotically synchronize their velocity, i.e.,
\[
\begin{bmatrix}
\dot{v}_1 \\
\dot{v}_2
\end{bmatrix} = 
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} \tag{11}
\]
Thus, we build a system in which the two co-leaders take the control law (11) and followers take the control law (10). Then, the co-leader vehicle network is still able to asymptotically achieve a moving formation with the desired formation shape and a synchronized moving velocity.

**B. Formation shape control with determined scale**

From Theorem 3.1, it is known that the scale of the moving geometric formation $F_\xi$ depends on the the initial positions of the leaders according to (6). Thus, it is clear that if the two co-leaders know the formation scale and are able to control themselves to reach the desired distance between each other, then the whole group will asymptotically converge to the desired formation with determined scale. To this end, the two co-leaders need to access the relative position of each other. A local control law is borrowed from [3] for the two co-leaders, i.e.,
\[
\begin{align}
\dot{z}_1 &= v_0 + k \|z_2 - z_1\|^2 - r_{12} \eta_1, \\
\dot{z}_2 &= v_0 + k \|z_1 - z_2\|^2 - r_{12} \eta_2
\end{align} \tag{12a}
\]
where $k \in \mathbb{R}^+$ is a scalar parameter to adjust the velocity, $v_0$ is a synchronized constant velocity and $r_{12}$ is the desired distance between two leaders. As discussed in Remark 3.2, if the two co-leaders do not have a synchronized velocity, they can do so by adopting the consensus protocol.

The control law for the followers remains the same as for formation shape control with undetermined scale, i.e.,
\[
\begin{align}
\dot{z}_i &= d_i \sum_{j \in N_i(G)} w_{ij} (z_j - z_i) + \eta_i, \\
\dot{\eta}_i &= \sum_{j \in N_i(\mathcal{H})} \alpha_{ij} (\eta_j - \eta_i), \quad i = 3, \cdots, n
\end{align} \tag{12b}
\]
where $w_{ij}$ is a complex weight of the complex-valued Laplacian matrix $L$ so that $L \xi = 0$ holds, $d_i \in \mathbb{C}$ is a control parameter to be designed later for the purpose of stability, $\alpha_{ij}$ is an arbitrary positive number of the real-valued Laplacian matrix $H$, $N_i(G)$ and $N_i(\mathcal{H})$ are the in-neighbor sets of the agent $i$ in the sensing graph $G$ and the communication graph $\mathcal{H}$, and $\eta_i$ is the velocity estimate of agent $i$.

Let $\eta_1 = \eta_2 = v_0$ and thus $\dot{\eta}_1 = \dot{\eta}_2 = 0$ since $v_0$ is a constant. Let $z_1 = [z_1, z_2]^T$, $z_f = [z_3, \ldots, z_n]^T$, $\eta_1 = [\eta_1, \eta_2]^T$ and $\eta_f = [\eta_3, \ldots, \eta_n]^T$. Then the whole system can be written in the vector form
\[
\begin{bmatrix}
\dot{z}_f \\
\dot{\eta}_f
\end{bmatrix} =
A
\begin{bmatrix}
z_f \\
\eta_f
\end{bmatrix} + B
\begin{bmatrix}
z_1 \\
v_0
\end{bmatrix} \tag{13a}
\]
\[
\begin{bmatrix}
\dot{z}_i \\
\dot{\eta}_i
\end{bmatrix} =
A
\begin{bmatrix}
z_f \\
\eta_f
\end{bmatrix} + B
\begin{bmatrix}
z_i \\
v_0
\end{bmatrix} \tag{13b}
\]
where 
\[ g(z_t) = \begin{bmatrix} v_0 + k(z_2 - z_1)(\|z_2 - z_1\|^2 - r_2^2) \\ v_0 + k(z_1 - z_2)(\|z_1 - z_2\|^2 - r_1^2) \end{bmatrix} \]
\[ A = \begin{bmatrix} -D_f L_{ff} & I_n \\ 0 & -H_{ff} \end{bmatrix}, \text{ and } B = \begin{bmatrix} -D_f L_{ff} \\ -H_{ff} \end{bmatrix} \]

with \(D_f = \text{diag}\{d_3, \ldots, d_n\} \).

Then we have the following result.

**Theorem 3.3:** Suppose \(\xi \in \mathbb{C}^n\) satisfying \(\xi \neq \xi_j\). If \(L\xi = 0\), \(\det(L_{ff}) \neq 0\) and \(\det(H_{ff}) \neq 0\), then the system (13) has two steady solutions: One is \(z^*(t) = (c_1 + v_0 t)1_n + e^{\phijo(\|\xi\|^2/2)}\xi\) and \(\eta_f^* = v_0 1_{(n-2)}\), where \(\xi_2 = \|\xi_1 - \xi_2\|\), and \(c_1 \in \mathbb{C}\) and \(\phi\) depend on the initial condition. The other is the trivial solution \(z^*(t) = (c' + v_0 t)1_n\) and \(\eta_f^* = v_0 1_{(n-2)}\).

**Remark 3.3:** The first steady solution of system (13) in Theorem 3.3 corresponds to the desired formation with a determined scale and the other trivial solution corresponds to the state consensus. As we will see later, the trivial solution of consensus is not stable while the first steady solution corresponding to the desired moving formation is asymptotically stable.

**Proof of Theorem 3.3:** Similar to the proof of Theorem 3.1, it is not hard to see that the steady solution of system (13) must have the form
\[ z^*(t) = (c_1 + v_0 t)1_n + c_2 \xi \text{ and } \eta_f^* = v_0 1_{(n-2)} \]

Moreover, note from the local control law (12) that at steady states, either \(z_1^*(t) = z_2^*(t)\) or \(|z_2^*(t) - z_1^*(t)| = r_{12}\). Therefore, the conclusion follows by taking into account the above two observations.

Next we present the stability result for the closed-loop system under the proposed control strategy.

**Theorem 3.4:** Suppose that the conditions in Theorem 3.3 hold and also \(z_1(0) \neq z_2(0)\). If \(D_f\) can assign all the eigenvalues of \(D_f L_{ff}\) in the open right complex plane, then the system (13) asymptotically reaches a moving geometric formation with a determined scale as described in Theorem 3.3.

Before proving the theorem, we introduce several notions and results for cascade systems.

The system (13) can be described as a cascade system (Fig. 2). The leaders’ subsystem, \(A_s\), is an autonomous system, while the followers’ subsystem, \(B_s\), is a nonautonomous system. The output \(z_1\) of \(A_s\) is the input of \(B_s\).

![Fig. 2. A cascade system representation.](image)

The notion of input-to-state stability is presented in the following.

**Definition 3.1 ([5]):** Consider a nonlinear dynamical system
\[ \dot{x} = f(t, x, u) \]
with the state \(x(t)\) and input \(u(t)\). It is said to be input-to-state stable if for every initial state \(x(t_0) \in \mathbb{R}^n\) and every continuous and bounded input \(u(t) \in \mathbb{R}^m\), the solution \(x(t)\) exists for all \(t > t_0\) and satisfies
\[ \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|), \]
where \(\beta(s, t)\) is a class \(\mathcal{K}\) function and \(\gamma(s)\) is a class \(\mathcal{K}\) function.

**Lemma 3.2 ([5]):** For a cascade system, if the subsystem \(\dot{x}_1 = f_1(t, x_1, x_2)\), with \(x_2\) as input, is input-to-state stable and the origin of \(\dot{x}_2 = f_2(t, x_2)\) is globally uniformly asymptotically stable, then the origin of the cascade system \(\dot{x}_1 = f_1(t, x_1, x_2)\) and \(\dot{x}_2 = f_2(t, x_2)\) is globally uniformly asymptotically stable.

Finally, we present a result here about the asymptotic behavior of the two co-leaders.

**Theorem 3.5 ([31]):** For two co-leader agents with the control law (12), if \(z_1(0) \neq z_2(0)\), then
\[ \|z_1(t) - z_2(t)\| \to r_{12} \]
\[ z_1(t) \to v_0 \]
\[ z_2(t) \to v_0 \]
exponentially as \(t \to \infty\).

**Proof of Theorem 3.4:** First, consider the subsystem \(B_s\) in Fig. 2, which can be re-written in the following form
\[ \dot{x} = Ax + Bu. \]

By the condition that \(D_f\) can assign the eigenvalues of \(D_f L_{ff}\) in the open right complex plane, it follows that \(A\) is Hurwitz. So \(\|e^{(t-t_0)A}\| \leq e^{-\rho(t-t_0)}\) for some positive \(\rho\). Substituting this inequality into the solution of (16)
\[ x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(\tau-t)A}B u(\tau)d\tau, \]
we then obtain
\[ \|x(t)\| \leq e^{-\rho(t-t_0)} \|x(t_0)\| + \int_{t_0}^{t} e^{-\rho(t-t')} \|B\| \|u(\tau)\| d\tau \]
\[ \leq e^{-\rho(t-t_0)} \|x(t_0)\| + \frac{\|B\|}{\rho} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \]
Note that \(ke^{-\rho(t-t_0)}\) \(\|x(t_0)\|\) is a class \(\mathcal{K}\) function of \(\|x(t_0)\|\) and \(t - t_0\), and \(\frac{\|B\|}{\rho} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\) is a class \(\mathcal{K}\) function of \(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\). It follows that \(B_s\) is input-to-state stable, according to the Definition 3.1.

On the other hand, by Theorem 3.5, the subsystem \(A_s\) is globally exponentially stable except the initial condition \(z_1(0) = z_2(0)\).

Therefore, applying Lemma 3.2, the cascade system \(A_s\) and \(B_s\) is globally asymptotically stable if \(z_1(0) \neq z_2(0)\).

Thus, the conclusion follows.

**C. Collision avoidance**

In this subsection, we address the collision avoidance issue in formation control. To this end, we add extra control \(u_i(t)\) to the formation control law, which only takes effect when some neighbors come into its collision-avoidance region. Here we only discuss the case of formation control with undetermined scales. The other case with determined scales...
for formation control will be the same by adding this extra collision avoidance control.

With this extra collision avoidance control, the overall system becomes

$$\begin{bmatrix} \dot{z} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -DL & I_n \\ 0 & -H \end{bmatrix} \begin{bmatrix} z \\ \eta \end{bmatrix} + \begin{bmatrix} u \end{bmatrix}$$

(17)

where $u = [u_1, u_2, \ldots, u_n]^T$.

Next we discuss how to design the collision avoidance control $u$. Towards this objective, we assume that each agent $i$ can locate nearby obstacles and other agents inside a collision avoidance region (Fig. 3), which is defined as follows.

$$D_i = \{ p \in \mathbb{C} : r < \|z_i - p\| \leq R \}$$

where $R \in \mathbb{R}$ is the avoidance radius and $r \in \mathbb{R}$ is the radius the collision must occur, which is the smallest safe distance from any other agent or obstacle. In other words, when the distance is less than $r$, it is called the dead region, which is defined as

$$\Omega_i = \{ p \in \mathbb{C} : \|z_i - p\| \leq r \}.$$

Inspired by the work [10], we consider the following potential function

$$P_{ij}(z_i, z_j) = \left( \min \left\{ 0, \frac{\|z_i - z_j\|^2 - R^2}{\|z_i - z_j\|^2 - r^2} \right\} \right)^2, \quad z_j \neq z_i$$

where $z_j \in D_i$.

Thus, it is known that if $\|z_i - z_i\| \geq R$ or $\|z_i - z_j\| \leq r$, then

$$\frac{\partial P_{ij}}{\partial z_i} = 0.$$

If $R > \|z_i - z_j\| > r$, then

$$\frac{\partial P_{ij}}{\partial z_i} = 4 \frac{(R^2 - r^2)(\|z_i - z_j\|^2 - R^2)}{(\|z_i - z_j\|^2 - r^2)^3} (z_i - z_j).$$

And if $\|z_i - z_j\| = r$, then $\frac{\partial P_{ij}}{\partial z_i}$ is not defined.

The collision avoidance control is given below,

$$u_i = -\sum_{z_j \in D_i} \frac{\partial P_{ij}}{\partial z_i}.$$  

(18)

**Theorem 3.6:** For the dynamic system (17) with $u$ given in (18). If $\|z_i(0) - z_j(0)\| > r$ for all $i$ and $j$, then $\|z_i(t) - z_j(t)\| > r$ for all $i, j$ and for all $t \in (0, \infty)$.

**Proof:** Suppose by contradiction that there exist a pair $i$ and $j$ such that at time $t_1 > 0,$

$$V_{ij}(t_1) := \|z_i(t_1) - z_j(t_1)\|^2 = r^2.$$

Then it is certain that there exists a time $t' < t_1$, very close to $t_1$, such that

$$V_{ij}(t') = \|z_i(t') - z_j(t')\|^2 = r^2 + \varepsilon$$

with a very small $\varepsilon > 0$ and $\dot{V}_{ij}(t') < 0$.

Considering the control law (17), it is not hard to show that when $\|z_i(t') - z_j(t')\|^2 = r^2 + \varepsilon$ with $\varepsilon$ arbitrarily small,

$$\dot{V}_{ij}(t') = 2(\dot{z}_i - \dot{z}_j)(\dot{z}_i - \dot{z}_j) > 0$$

since $2(\dot{z}_i - \dot{z}_j)(\dot{z}_i - \dot{z}_j)$ tends to $\infty$ as $\varepsilon$ approaches $0$. Therefore, we reach a contradiction. 

**IV. SIMULATIONS**

In this section, we present several simulations to illustrate our results.

A. **Formation shape control with undetermined scale**

Consider a system consisting of five agents where agent 1 and 2 are two leaders, and agent 3, 4, and 5 are followers. The sensing graph $\mathcal{G}$ is given in Fig. 4. The communication graph $\mathcal{H}$ is different from the sensing graph $\mathcal{G}$ and it is shown in Fig. 5.

Consider a geometric formation with a formation vector (Fig. 4) defined as

$$\xi = \begin{bmatrix} 2t & 2 & 1 - 2t & -1 - 2t & -2 \end{bmatrix}^T.$$

The complex-valued Laplacian matrix $L$ of the sensing graph and real-valued matrix $H$ of the communication graph take

![Fig. 3. Avoidance (radius R) and dead (radius r) regions.](image)

![Fig. 4. The sensing graph G and a formation vector ξ.](image)

![Fig. 5. The communication graph H.](image)
the following forms
\[
L = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2 & -3 - 2\zeta & 1 + 2\zeta \\
0 & 0 & 5 & -7 - 4\zeta & 2 + 4\zeta \\
5 & 0 & 0 & 2 - 6\zeta & -7 + 6\zeta
\end{bmatrix},
\]
and
\[
H = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
-1 & -1 & 0 & 2 & 0 \\
-1 & 0 & 0 & -1 & 2
\end{bmatrix}.
\]
The following
\[
D = \text{diag}\{1, 1, -1, -1, -1\}
\]
is used to assign the eigenvalues of the closed-loop system (9) at \(-8.0523 + 2.8475\zeta, -8.1341 - 2.5424\zeta, -0.8136 - 0.305\zeta, -2, -2, -2, 0, 0, 0, 0\) to ensure the asymptotic stability.

Consider that the two leaders have the synchronized velocity \(v_0 = 2 + \iota\). The simulation results are shown in Fig. 6. Fig. 6(a) shows the trajectories of the five agents which asymptotically converge to a moving geometric formation. The small circles represent the initial positions of the agents, while the colored dots represent the current positions in the steady state. Fig. 6(b) plots the evolution of the components of \(Lz\). As we can see, all the components converge to zero, which also means the states of the five agents approaches the null space of \(L\) (or equivalently to say, reaching the desired formation).

### B. Formation shape control with determined scale

Under the same multi-agent system, suppose the two co-leaders take the control law (12a) for the purpose of scaling control. A simulation result is presented in Fig. 7, in which the system first achieves a moving geometric formation and then switch to a formation with a small scale in order to pass a narrow corridor.

### C. Collision avoidance

To test the collision avoidance strategy, we run both formation control with or without the extra collision avoidance control term for the same system. Fig. 8(a) records the trajectories of the five agents in the process of achieving a desired formation without the extra collision avoidance control term, while the trajectories in Fig. 8(b) are produced under the control law with the collision avoidance control term. It can be seen that in Fig. 8(a), agent 1 and 5 might have a collision while there are no collision between them in Fig. 8(a). This can be seen more clearly from Fig.9, which shows the relative distance between agent 1 and 5. From Fig.9 we notice that there is a time at which the relative distance between agent 1 and 5 is below the safety threshold for the control without the collision avoidance strategy, while with the help of the collision avoidance strategy, the collision is avoided with the relative distance always larger than \(r\).
V. CONCLUSIONS AND FUTURE WORK

In this paper, we present local formation control strategies with undetermined and determined formation scales for co-leader vehicle networks. An auxiliary dynamic system is introduced for each agent to store and update the estimate of its own velocity according to the information from its neighbors in the communication network. Then the formation control runs successfully without requiring all the followers to know the leaders’ velocity. Moreover, we obtain that a network of agents can almost globally asymptotically converge to a desired formation shape by using tools from linear system theory and input-to-state stability theory. Finally, an extra collision avoidance control law is introduced to guarantee collision avoidance, which takes effect only when two agents come too closer and so does not affect the evolution towards the desired formation. However, the extra collision avoidance control input may lead to some undesirable equilibrium points. This is left for future study. Moreover, as a future work, our control approach can be extended to multi-agent systems with more complex individual dynamics.

REFERENCES


